Differentiation of measures

Definition: Let μ & λ be locally finite Borel measures on $\mathbb{R}^d$.
The upper & lower derivatives of μ with respect to λ at $x \in \mathbb{R}^d$ are

$$\overline{D}(\mu,\lambda,x) := \overline{\lim_{r \downarrow 0}} \frac{\mu(B(x,r))}{\lambda(B(x,r))}$$

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Here $\frac{0}{0}$ is meant to be 0

At points x where the limit exists we define the derivative of μ w.r.t. λ by

$D(\mu,\lambda,x) = \overline{D}(\mu,\lambda,x) = \underline{D}(\mu,\lambda,x)$.

Home work: Prove that the functions $x \mapsto \overline{D}(\mu,\lambda,x)$ & $x \mapsto \underline{D}(\mu,\lambda,x)$ are Borel measurable functions in the special case when $\lambda = \mathrm{Leb}_d$ via the following steps:

(a) Show that $\overline{\lim_{i \to \infty}} \mu(B(x_i,r)) \le \mu(B(x,r))$ if $x_i \to x$. (That is $\boxed{x \mapsto \mu(B(x,r))}$ is upper semicontinuous.)

(b) Prove that the upper and lower limits in $\overline{D}(\mu,\lambda,x)$ and $\underline{D}(\mu,\lambda,x)$ do not change if r is restricted to positive rationals. To see this you can use that
– $\mu(B(x,r))$ is monoton in r,
– $\mathrm{Leb}_d(B(x,r))$ is continuous in r.

(c) Use that the infimum & supreme of a sequence of Borel functions is a Borel function.

Definition: Let μ and λ be measures on $\mathbb{R}^d$. We say that $\boxed{\mu \ll \lambda}$, μ is absolute continuous w.r.t. λ if $\boxed{\lambda(A) = 0 \Rightarrow \mu(A) = 0,\ \forall A}$

Theorem (differentiation of μ w.r.t. λ) Let μ and λ be Radon measures.

(1) For λ a.a. $x \in \mathbb{R}^d$ $D(\mu,\lambda,x)$ exists & finite.

(2) Let $B \subset \mathbb{R}^d$ be Borel. Then

$$\int_B D(\mu,\lambda,x)\, d\lambda(x) \le \mu(B) \quad (*)$$

with equality if $\mu \ll \lambda$.

(3) $\mu \ll \lambda$ if and only if $\underline{D}(\mu,\lambda,x) < \infty$ for μ a.a. $x \in \mathbb{R}^d$.

To prove this theorem we need a lemma.

Lemma: Let μ, λ be Radon measures on $\mathbb{R}^d$, $0 < t < \infty$ and $A \subset \mathbb{R}^d$.

(1) $\forall x \in A$, $\underline{D}(\mu,\lambda,x) \le t \Rightarrow \mu(A) \le t\lambda(A)$,

(2) $\forall x \in A$, $\overline{D}(\mu,\lambda,x) \ge t \Rightarrow \mu(A) \ge t\lambda(A)$.

Proof of the Lemma:

Now we prove (1). Let $\varepsilon > 0$. Choose an open set U s.t.

$A \subset U$ & $\lambda(U) < \lambda(A) + \varepsilon$.

We apply the theorem called "Vitali CT for Radon measures" for the measure μ as follows:

Let $\mathcal{B}$ be the collection of balls defined as follows: For every $x \in A$ pick a closed ball B_x centered at x satisfying:

$$B_x \subset U, \quad \mu(B_x) < (t+\varepsilon)\lambda(B_x) \quad (**)$$

Let $\mathcal{B}$ be the collection of balls $\mathcal{B} = \{B_x\}_{x \in A}$. Apply the above mentioned CT for $\mathcal{B}$, A and μ.

We get the sequence of closed balls $\{B_i\}_{i=1}^{\infty}$ s.t.
$$\begin{cases} B_i \subset U, \\ \mu(B_i) < (t+\varepsilon)\lambda(B_i), \\ \mu\left(A \setminus \bigcup_i B_i\right) = 0. \end{cases}$$

Then

$$\mu(A) \le \sum_i \mu(B_i) = (t+\varepsilon)\sum_i \lambda(B_i) \le (t+\varepsilon)\lambda(U) \le (t+\varepsilon)(\lambda(A)+\varepsilon).$$

Letting $\varepsilon \downarrow 0$, we obtain: $\mu(A) \le t\lambda(A)$. This proves (1). The proof of (2) is Home Work! (It can be proved in the same way.) ∎

Now we can prove the differentiation of measure theorem above.

Proof of the differentiation of measures thm.

For $0 < r < \infty$, $0 < s < t < \infty$ let

$A_{s,t,r} := \{x \in B(r) : \underline{D}(\mu,\lambda,x) \le s < t \le \overline{D}(\mu,\lambda,x)\}$,

$A_{u,r} := \{x \in B(r) : \overline{D}(\mu,\lambda,x) \ge u\}$.

Using the lemma above:

$t\lambda(A_{s,t,r}) \le \mu(A_{s,t,r}) \le s\lambda(A_{s,t,r}) < \infty$,

$u\lambda(A_{u,r}) \le \mu(A_{u,r}) \le \mu(B(r)) < \infty$.

The first inequality implies that $\lambda(A_{s,t,r}) = 0$ since $s < t$. Further, the second inequality yields that

$$\lambda\Big(\bigcap_{u>0} A_{u,r}\Big) = \lim_{u \to \infty} \lambda(A_{u,r}) = 0.$$

$G := \{x : \exists D(\mu,\lambda,x) < \infty\}$.

Then

$$G^c = \bigcup_{\substack{s,t \in \mathbb{Q}^+ \\ r \in \mathbb{N}^+}} A_{s,t,r} \cup \bigcap_{u>0} A_{u,r}$$

So, $\lambda(G^c) = 0$ which completes the proof of (1).

$\boxed{\text{Now we prove (2)}}$ Let $1 < t < \infty$.

$B_p := \{x \in B : t^p \le D(\mu,\lambda,x) < t^{p+1}\}$, $p = 0, \pm 1, \pm 2, \pm 3 \ldots$

Now we use part (1) of this thm & Part (2) of the previous lemma:

$$\int_B D(\mu,\lambda,x)\, d\lambda(x) = \sum_{p=-\infty}^{\infty} \int_{B_p} D(\mu,\lambda,x)\, d\lambda(x)$$

$$\le \sum_{p=-\infty}^{\infty} t^{p+1}\lambda(B_p) \le t \sum_{p=-\infty}^{\infty} \mu(B_p)$$

$\le t \cdot \mu(B)$. Let $t \downarrow 1$. Then

$$\int_B D(\mu,\lambda,x)\, d\lambda(x) \le \mu(B).$$

If $\mu \ll \lambda$ then any λ-measure zero set is also a μ-measure zero set. Hence

$D(\mu,\lambda,x) = D(\lambda,\mu,x)^{-1} > 0$ for μ-a.a. x

$\mu(B) = \sum_{p=-\infty}^{\infty} \mu(B_p)$. A similar argument as above gives that $\int_B D(\mu,\lambda,x)\, d\lambda(x) \ge \mu(B)$ Home Work: Complete the proof.

$\boxed{\text{Now we prove (3)}}$ Assume that

$\underline{D}(\mu,\lambda,x) < \infty$ for μ- a.a. $x \in \mathbb{R}^d$.

Let $A \subset \mathbb{R}^d$ s.t. $\lambda(A) = 0$. Using the previous lemma:

$$\mu(\underbrace{\{x \in A : \underline{D}(\mu,\lambda,x) \le u\}}_{A_u}) \le u \cdot \lambda(A) = 0.$$

By our assumption

$$\mu\left(\mathbb{R}^d \setminus \{x : \underline{D}(\mu,\lambda,x) < \infty\}\right) = 0.$$

So,

$$\mu\Big(\mathbb{R}^d \setminus \bigcup_{u=1}^{\infty} A_u\Big) = 0.$$

On the other hand we saw that $\mu(A_u) = 0$. That is $\mu(A) = 0$.

In this way we have proved that

$\lambda(A) = 0 \Rightarrow \mu(A) = 0$

Which means that $\mu \ll \lambda$.

This completes the proof of differentiation of measures theorem. ∎

Corollary

(1) Let $A \subset \mathbb{R}^d$ be λ-measurable. Then

$$\lim_{r \downarrow 0} \frac{\lambda(A \cap B(x,r))}{\lambda(B(x,r))} = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in A^c \end{cases}$$

holds for λ- a.a. $x \in \mathbb{R}^d$.

(2) Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty, \infty\}$ be locally integrable w.r.t. λ.
($\forall x, \exists r$, $\int_{B(x,r)} f(x)\, d\lambda(x)$ is finite)
Then for λ- a.a. $x \in \mathbb{R}^d$:

$$\lim_{r \downarrow 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f(x)\, d\lambda(x) = f(x). \quad (**)$$

Proof of the Corollary:

(2) $\Rightarrow$ (1) with $f = \mathbb{1}_A$. So now we prove (2).

We may assume that $f \ge 0$.

Let $\mu(A) := \int_A f(x)\, d\lambda(x)$.

Clearly $\mu \ll \lambda$ and from part (2) of the previous thm:

$$(*) \quad \int_B D(\mu,\lambda,x)\, d\lambda(x) = \mu(B) = \int_B f(x)\, d\lambda(x),$$

for Borel sets $B \subset \mathbb{R}^d$. Clearly, $(*)$ implies that $f(x) = D(\mu,\lambda,x)$ holds for λ-a.a. x. ∎

Remarks: (1) If $\lambda = \mathrm{Leb}_d$ then

$$(*) \quad \overline{\lim_{\delta \downarrow 0}} \left\{ \left| \mathrm{Leb}_d(B)^{-1} \cdot \int_B f(x)\,dx - f(x) \right| : B \text{ is a ball}, x \in B, |B| < \delta \right\} = 0.$$

Home Work: Prove $(*)$ in such a way that use the Vitali CT for Leb. measure instead of the Vitali CT for Radon measure. (The latter one was used in the differentiation of measures thm above. This thm does not imply $(*)$ but we get $(*)$ from an analogous proof with Covering CT for the Leb. measure instead of CT for the Radon measures.)

(2) The formula $(**)$ in Corollary (2) can be stregthened to:

$$\lim_{r \downarrow 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, d\lambda(y) = 0.$$

holds for λ- a.a. $x \in \mathbb{R}^d$. To see this apply the second part of the Corollary to the function $f(x) - q$ for rational q.

Definition: We say that the Radon measures λ & μ are singular ($\lambda \perp \mu$) if $\exists A \subset \mathbb{R}^d$ s.t.

$\lambda(A) = 0$ & $\mu(\mathbb{R}^d \setminus A) = 0$.

Now we prove a combination of the Radon-Nikodim thm & Leb. decomposition thm.

Theorem (M.2.17): Let μ, λ be finite Radon measures on $\mathbb{R}^d$. Then $\exists$ a Borel function f and a Radon measure ν s.t.

(1) $\lambda \perp \nu$.

(2) $\mu(B) = \int_B f\, d\lambda + \nu(B)$, $\forall B \subset \mathbb{R}^d$ Borel.

(3) $\mu \ll \lambda \iff \nu = 0$.

Proof: $A := \{x \in \mathbb{R}^d : \underline{D}(\mu,\lambda,x) < \infty\}$,

$\mu_1 := \mu|_A$ & $\nu := \mu|_{\mathbb{R}^d \setminus A}$.

Then $\mu = \mu_1 + \nu$. Further, $\lambda \perp \nu$ by the first part of the theorem

$\mu_1 \ll \lambda$ is clear. So, $\mu_1(B) = \int_B D(\mu,\lambda,x)\, d\lambda(x)$.

(3) is obvious. ∎